SOLVING $a \pm b = 2c$ IN THE ELEMENTS OF FINITE SETS

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Abstract. We show that if A and B are finite sets of real numbers, then the number of triples $(a,b,c)\in A\times B\times(A\cup B)$ with $a+b=2c$ is at most $(0.15+o(1))(|A|+|B|)^2$ as $|A|+|B|\to\infty$. As a corollary, if A is antisymmetric (that is, $A\cap(-A)=\varnothing$), then there are at most $(0.3+o(1))|A|^2$ triples (a,b,c) with $a,b,c\in A$ and $a-b=2c$. In the general case where A is not necessarily antisymmetric, we show that the number of triples (a,b,c) with $a,b,c\in A$ and $a-b=2c$ is at most $(0.5+o(1))|A|^2$. These estimates are sharp.

1. Introduction and summary of results

For a finite real set A of given size, the number of three-term arithmetic progressions in A is maximized when A itself is an arithmetic progression. This follows by observing that for any integer $1\le k\le |A|$, the number of three-term progressions in A with the middle term at the kth largest element of A is at most $\min\{k-1,|A|-k\}$. A simple computation leads to the conclusion that the number of triples $(a,b,c)\in A\times A\times A$ with $a+b=2c$ is at most $0.5|A|^2+0.5$.

Suppose now that only those progressions with the least element below, and the greatest element above the median of A, are counted; what is the largest possible number of such "scattered" progressions? This problem was raised in [NPPZ] in connection with a combinatorial geometry question by Erdős. Below we give it a complete solution; indeed, we solve a more general problem, replacing the sets of all elements below / above the median with arbitrary finite sets.

Theorem 1. *If A and B are finite sets of real numbers, then the number of triples (a,b,c) with $a\in A,\ b\in B,\ c\in A\cup B$, and $a+b=2c$, is at most* $0.15(|A|+|B|)^2+0.5(|A|+|B|)$.

For a subset A of an abelian group, write $-A:=\{-a\colon a\in A\}$. We say that A is *antisymmetric* if $A\cap(-A)=\varnothing$. Thus, for instance, any set of positive real numbers is antisymmetric.

For an antisymmetric set A, the number of triples (a,b,c) with $a\in A,\ b\in -A,\ c\in A\cup(-A)$, and $a+b=2c$, is twice the number of triples (a,b,c) with $a,b,c\in A$ and $a-b=2c$. Hence, Theorem 1 yields

Corollary 1. *If A is a finite antisymmetric set of real numbers, then the number of triples (a,b,c) with $a,b,c \in A$ and $a-b=2c$ is at most $0.3|A|^2+0.5|A|$.*

The following example shows that the coefficient 0.3 of Corollary 1, and therefore also the coefficient 0.15 of Theorem 1, is best possible.

Example. Fix an integer $m \ge 1$, and let A consist of all positive integers up to m, and all even integers between m and $4m$ (taking all odd integers will do as well). Assuming for definiteness that m is even, we thus can write

$$A = [1,m] \cup \{m+2, m+4, \dots, 4m\}.$$

Notice, that A contains $m/2$ odd elements and $2m$ even elements, of which exactly m are divisible by 4; in particular, $|A| = 5m/2$. For every triple $(a,b,c) \in A \times A \times A$ with $a-b=2c$, we have $a \equiv b \pmod 2$ and $a > b$. There are $\binom{m/2}{2}$ such triples with a and b both odd, and $2\binom{m}{2}$ triples with a and b both even and satisfying $a \equiv b \pmod 4$. Furthermore, it is not difficult to see that there are $\frac{3}{4}m^2$ triples with a and b both even and satisfying $a \not\equiv b \pmod 4$. Thus, the total number of triples under consideration is

$$\binom{m/2}{2} + 2\binom{m}{2} + \frac{3}{4}m^2 = \frac{15}{8}\,m^2 - \frac{5}{4}\,m = \frac{3}{10}\,|A|^2 - \frac{1}{2}\,|A|,$$

the first summand matching the main term of Corollary 1.

Our second principal result addresses the same equation as Corollary 1, but in the general situation where the antisymmetry assumption got dropped.

Theorem 2. *If A is a finite set of real numbers, then the number of triples (a,b,c) with $a,b,c \in A$ and $a-b=2c$ is at most $0.5|A|^2+0.5|A|$.*

The main term of Theorem 2 is best possible as it is easily seen by considering the set $A=[-m,m]$, where $m \ge 1$ is an integer. For this set, the number of triples $(a,b,c) \in A \times A \times A$ with $a-b=2c$ is equal to the number of pairs $(a,b) \in A \times A$ with a and b of the same parity, which is $(m+1)^2+m^2 = 0.5|A|^2+0.5$.

It is a challenging problem to generalize our results and investigate the equations $a \pm b = \lambda c$, for a fixed real parameter $\lambda > 0$. As it follows from [L98, Theorem 1], the number of solutions of this equation in the elements of a finite set of given size is maximized when $\lambda = 1$, and the set is an arithmetic progression, centered around 0. It would be interesting to determine the largest possible number of solutions for every *fixed* value of $\lambda \ne 1$, or at least to estimate the maximum over all positive $\lambda \ne 1$.

We remark that using a standard technique, our results extend readily onto finite subsets of torsion-free abelian groups. In contrast, extending Theorems 1 and 2 onto groups with a non-zero torsion subgroup, and in particular onto cyclic groups, seems

to be a highly non-trivial problem requiring an approach completely different from that used in the present paper.

In the next section we prepare the ground for the proofs of Theorems 1 and 2. The theorems are then proved in Sections 3 and 4, respectively.

2. The proofs: preparations

For finite sets A, B, and C of real numbers, let

$$T(A,B,C) := \big|\{(a,b,c) \in A \times B \times C \colon a+b=2c\}\big|.$$

We start with a simple lemma allowing us to confine to the integer case.

Lemma 1. *For any finite sets A and B of real numbers, there exist finite sets A' and B' of* integer *numbers with $|A'| = |A|$, $|B'| = |B|$ such that $T(A',B',A'\cup B') = T(A,B,A\cup B)$ and $T(A',-A',A') = T(A,-A,A)$.*

Proof. By the (weak version of the) standard simultaneous approximation theorem, there exist arbitrary large integer $q \ge 1$, along with an integer-valued function φ_q acting on the union $A \cup (-A) \cup B$, such that

$$\left| c - \frac{\varphi_q(c)}{q} \right| < \frac{1}{4q}, \quad c \in A \cup (-A) \cup B.$$

Let $A' := \varphi_q(A)$ and $B' := \varphi_q(B)$. It is readily verified that if q is large enough, then $|A'| = |A|$ and $|B'| = |B|$ and, moreover, an equality of the form $a \pm b = 2c$ with $a, b, c \in A \cup (-A) \cup B$ holds true if and only if $\varphi_q(a) \pm \varphi_q(b) = 2\varphi_q(c)$. The assertion follows. $\square$

Clearly, for finite sets of integers A, B, and C with $|C| \ge |A|+|B|$, the number of triples $(a,b,c) \in A \times B \times C$ satisfying $a+b=c$ can be as large as $|A||B|$. Our argument relies on the following lemma which improves this trivial bound in the case where $|C| < |A|+|B|$.

Lemma 2. *If A, B and C are finite sets of real numbers with $\max\{|A|,|B|\} \le |C| \le |A|+|B|$, then the number of triples $(a,b,c) \in A \times B \times C$ satisfying $a+b=c$ does not exceed*

$$|A||B| - \frac{1}{4}\left(|A|+|B|-|C|\right)^2 + \frac{1}{4}.$$

Proof. We use induction on $|A|+|B|-|C|$. The case where $|A|+|B|-|C| \le 1$ is immediate, and we thus assume that $|A|+|B|-|C| \ge 2$. If either A or B is empty, then the assertion is readily verified. Otherwise, we let $a_{\min} := \min A$ and $b_{\max} := \max B$, and observe that every $c \in C$ has at most one representation as $c = a_{\min} + b$ with $b \in B$, or of the form $c = a + b_{\max}$ with $a \in A$. Indeed, the same element $c \in C$ cannot have representations of both kinds simultaneously, unless they are identical: for,

$a_{\min} + b = a + b_{\max}$ yields $b - a = b_{\max} - a_{\min}$, whence $a = a_{\min}$ and $b = b_{\max}$. This shows that removing $a_{\min}$ form A, and simultaneously $b_{\max}$ from B, we loose at most $|C|$ triples $(a, b, c) \in A \times B \times C$ with $a + b = c$. Using now the induction hypothesis to estimate the number of such triples with $a \neq a_{\min}$ and $b \neq b_{\max}$, we conclude that the total number of triples under consideration is at most

$$|C| + (|A| - 1)(|B| - 1) - \frac{1}{4}\,(|A| + |B| - 2 - |C|)^2 + \frac{1}{4} \\ = |A||B| - \frac{1}{4}\,(|A| + |B| - |C|)^2 + \frac{1}{4}.$$

□

We note that Lemma 2 can also be deduced from the following proposition, which is a particular case of [L98, Theorem 1]; see [G32, HL28, HLP88] for earlier, slightly weaker versions.

For a finite set A of real numbers, write $\operatorname{mid}(A) := \frac{1}{2}\big(\min(A) + \max(A)\big)$.

Proposition 1. *Let A, B, and C be finite sets of integers. If A', B', and C' are blocks of consecutive integers such that $\operatorname{mid}(C')$ is at most 0.5 off from $\operatorname{mid}(A') + \operatorname{mid}(B')$, and $|A'| = |A|,\ |B'| = |B|,\ |C'| = |C|$, then the number of triples $(a, b, c) \in A \times B \times C$ with $a + b = c$ does not exceed the number of triples $(a', b', c') \in A' \times B' \times C'$ with $a' + b' = c'$.*

Loosely speaking, Proposition 1 says that the number of solutions of $a + b = c$ in the variables $a \in A,\ b \in B$, and $c \in C$ is maximized when A, B, and C are blocks of consecutive integers, located so that C captures the integers with the largest number of representations as a sum of an elements from A and an element from B. We leave it to the reader to see how Lemma 2 can be derived from Proposition 1.

We use Lemma 2 to estimate the quantity $T(A, B, C)$, which is the number of solutions of $a + b = c'$ with $a \in A,\ b \in B$, and $c' \in \{2c \colon c \in C\}$. It is also convenient to recast the estimate of the lemma in terms of the function G which we define as follows: if (ξ, η, ζ) is a non-decreasing rearrangement of the triple (x, y, z) of real numbers, then we let

$$G(x, y, z) := \begin{cases} \xi\eta & \text{if } \zeta \geq \xi + \eta, \\ \xi\eta - \frac{1}{4}\,(\xi + \eta - \zeta)^2 & \text{if } \zeta \leq \xi + \eta. \end{cases}$$

Thus, for instance, we have $G(9, 6, 7) = 38$, whereas $G(7, 14, 6) = 42$.

Corollary 2. *If A, B and C are finite sets of integers, then*

$$T(A, B, C) \leq G(|A|, |B|, |C|) + \frac{1}{4}.$$

We close this section with two lemmas used in the proofs of Theorems 1 and 2, respectively.

For real x, we let $x_+ := \max\{x, 0\}$ and use x_+^2 as an abbreviation for $(x_+)^2$.

Lemma 3. *For any real x, y, and z, we have*

$$G\Big(\frac{x+y}{2}, \frac{x+y}{2}, z\Big) = G(x,y,z) + \frac{1}{4}\,(x-y)^2 - \frac{1}{4}\,\big(|x-y|-z\big)_+^2.$$

Corollary 3. *For any real x, y, and z, we have*

$$G\Big(\frac{x+y}{2}, \frac{x+y}{2}, z\Big) \geq G(x,y,z).$$

Lemma 4. *If x and z are real numbers with $z \leq 2x$, then $G(x,x,z) \leq xz - \frac{1}{4}\,z^2$.*

To prove Lemma 3 one can assume $x \leq y$ (which does not restrict the generality) and verify the assertion in the four possible cases $z \leq x$, $x \leq z \leq (x+y)/2$, $(x+y)/2 \leq z \leq y$, and $z \geq y$. The proof of Lemma 4 goes by straightforward investigation of the two cases $x \leq z$ and $x \geq z$. We omit the details.

3. Proof of Theorem 1

We use induction on $|A| + |B|$.

By Lemma 1, we can assume that A and B are sets of integers. For $i, j \in \{0, 1\}$ let $A_i := \{a \in A \colon a \equiv i \pmod 2\}$ and $A_{ij} := \{a \in A \colon a \equiv i + 2j \pmod 4\}$, and define B_i and B_{ij} in a similar way. Also, write $m := |A|$, $m_i := |A_i|$, $m_{ij} := |A_{ij}|$, $n := |B|$, $n_i := |B_i|$, and $n_{ij} := |B_{ij}|$. Applying a suitable affine transformation to A and B, we can assume without loss of generality that $A \cup B$ contains both even and odd elements, and the total number of even elements in A and B is at least as large as the total number of odd elements:

$$0 < m_1 + n_1 \leq m_0 + n_0 < m + n. \tag{1}$$

Keeping the notation introduced at the beginning of Section 2, we want to estimate the quantity $T(A, B, A \cup B)$. Observing that $a + b = 2c$ implies that a and b are of the same parity, we write

$$T(A, B, A \cup B) = T(A_0, B_0, A_0 \cup B_0) + T(A_0, B_0, A_1 \cup B_1) + T(A_1, B_1, A \cup B) \tag{2}$$

and estimate separately each of the three summands in the right-hand side.

For the first summand, we notice that $a_0 + b_0 = 2c_0$ with $a_0 \in A_0$, $b_0 \in B_0$, and $c_0 \in A_0 \cup B_0$, implies that $a_0/2$ and $b_0/2$ are of the same parity. Hence, either $a_0 \in A_{00}$ and $b_0 \in B_{00}$, or $a_0 \in A_{01}$ and $b_0 \in B_{01}$, leading to the upper bound $m_{00}n_{00} + m_{01}n_{01}$. On the other hand, we can use induction (cf. (1)) to estimate the first summand by $0.15(m_0 + n_0)^2 + 0.5(m_0 + n_0)$. As a result,

$$T(A_0, B_0, A_0 \cup B_0) \leq \min\{0.15(m_0 + n_0)^2, m_{00}n_{00} + m_{01}n_{01}\} + 0.5(m_0 + n_0). \tag{3}$$

Similar parity considerations show that if $a_0 + b_0 = 2c_1$ with $a_0 \in A_0$, $b_0 \in B_0$, and $c_1 \in A_1 \cup B_1$, then either $a_0 \in A_{00}$ and $b_0 \in B_{01}$, or $a_0 \in A_{01}$ and $b_0 \in B_{00}$. Therefore, using Corollary 2, we get

$$\begin{aligned} T(A_0, B_0, A_1 \cup B_1) &= T(A_{00}, B_{01}, A_1 \cup B_1) + T(A_{01}, B_{00}, A_1 \cup B_1) \\ &\le G(m_{00}, n_{01}, m_1 + n_1) + G(m_{01}, n_{00}, m_1 + n_1) + 0.5. \end{aligned} \tag{4}$$

For the last summand in (2) we use the trivial estimate

$$T(A_1, B_1, A \cup B) \le m_1 n_1 \le 0.25(m_1 + n_1)^2. \tag{5}$$

Substituting (3)– (5) into (2), we get

$$\begin{aligned} T(A, B, A \cup B) \le \min\{&0.15(m_0 + n_0)^2, m_{00}n_{00} + m_{01}n_{01}\} \\ &+ G(m_{00}, n_{01}, m_1 + n_1) + G(m_{01}, n_{00}, m_1 + n_1) \\ &+ \frac{1}{4}(m_1 + n_1)^2 + 0.5(m_0 + n_0) + 0.5. \end{aligned} \tag{6}$$

Recalling (1), we estimate the remainder terms as

$$0.5(m_0 + n_0) + 0.5 \le 0.5(m + n).$$

To estimate the main term, for real x_0, x_1, y_0, y_1 we write

$$s := x_0 + x_1 + y_0 + y_1 \tag{7}$$

and let

$$\begin{aligned} f(x_0, x_1, y_0, y_1) := \min\{&0.15s^2, x_0y_0 + x_1y_1\} \\ &+ G(x_0, y_1, 1 - s) + G(x_1, y_0, 1 - s) \\ &+ 0.25(1 - s)^2. \end{aligned} \tag{8}$$

Remainder terms dropped, the right-hand side of (6) can then be written as $(m + n)^2 f(\xi_0, \xi_1, \eta_0, \eta_1)$, where

$$\xi_0 := \frac{m_{00}}{m + n}, \ \xi_1 := \frac{m_{01}}{m + n}, \ \eta_0 := \frac{n_{00}}{m + n}, \text{ and } \eta_1 := \frac{n_{01}}{m + n}.$$

With (1) in mind, we see that to complete the argument it suffices to prove the following lemma.

Lemma 5. *For the function f defined by (7)– (8), we have*

$$\max\{f(x_0, x_1, y_0, y_1) \colon x_0, x_1, y_0, y_1 \ge 0, \ 1/2 \le s \le 1\} \le 0.15.$$

The inequality of Lemma 5 is surprisingly delicate, and the proof presented in the remaining part of this section is rather tedious. The reader trusting us about the proof may wish to skip on to Section 4, where the proof of Theorem 2 (independent of Theorem 1) is given.

Proof of Lemma 5. Since $f(x_0, x_1, y_0, y_1) = f(y_0, y_1, x_0, x_1)$, switching, if necessary, x_0 with y_0, and x_1 with y_1, we can assume that

$$x_0 + x_1 \geq y_0 + y_1. \tag{9}$$

Similarly, $f(x_0, x_1, y_0, y_1) = f(x_1, x_0, y_1, y_0)$ shows that x_0 can be switched with x_1, and y_0 with y_1 to ensure that

$$x_0 + y_0 \geq x_1 + y_1. \tag{10}$$

(Observe, that switching x_0 with x_1 and y_0 with y_1 does not affect (9).) Thus, from now on we assume that (9) and (10) hold true.

Our big plan is to investigate the effect made on f by replacing the variables x_0 and y_1 with their average $(x_0 + y_1)/2$, and, simultaneously, replacing the variables x_1 and y_0 with their average $(x_1 + y_0)/2$. We show that either

$$f\Big(\frac{x_0 + y_1}{2}, \frac{x_1 + y_0}{2}, \frac{x_1 + y_0}{2}, \frac{x_0 + y_1}{2}\Big) \geq f(x_0, x_1, y_0, y_1) \tag{11}$$

(meaning that f is non-decreasing under such "balancing"), or

$$x_0 \geq y_1 + (1 - s), \tag{12}$$

$$y_0 \geq x_1 + (1 - s), \tag{13}$$

and

$$3(x_0 + y_0) + (x_1 + y_1) \geq 2. \tag{14}$$

In both cases, the problem reduces to maximizing a function in just two variables.

We thus assume that (11) fails, aiming to prove that (12)– (14) hold true. Along with (8) and Corollary 3, our assumption implies

$$\frac{1}{2}(x_0 + y_1)(x_1 + y_0) < x_0 y_0 + x_1 y_1,$$

simplifying to

$$(x_0 - y_1)(x_1 - y_0) < 0.$$

Writing (10) as $x_0 - y_1 \geq x_1 - y_0$, we conclude that

$$x_0 > y_1 \text{ and } y_0 > x_1 \tag{15}$$

(which the reader may wish to compare with (12) and (13)).

Let

$$\mathcal{O} := x_0 y_0 + x_1 y_1 + G(x_0, y_1, 1 - s) + G(x_1, y_0, 1 - s)$$

and

$$\mathcal{N} := \frac{1}{2}(x_0 + y_1)(x_1 + y_0) + G\Big(\frac{x_0 + y_1}{2}, \frac{x_0 + y_1}{2}, 1 - s\Big) + G\Big(\frac{x_1 + y_0}{2}, \frac{x_1 + y_0}{2}, 1 - s\Big)$$

(the script letters standing for "old" and "new"); thus, $\mathcal{N} < \mathcal{O}$ by the assumption that (11) fails, (8), and Corollary 3. From Lemma 3 and (15) we get

$$\begin{aligned}\mathcal{N} - \mathcal{O} &= \frac{1}{2}\,(x_0 - y_1)(x_1 - y_0) + \frac{1}{4}\,(x_0 - y_1)^2 - \frac{1}{4}\,(|x_0 - y_1| - (1-s))_+^2 \\ &\qquad\qquad\qquad + \frac{1}{4}\,(x_1 - y_0)^2 - \frac{1}{4}\,(|x_1 - y_0| - (1-s))_+^2 \\ &= \frac{1}{4}\,(x_0 + x_1 - y_0 - y_1)^2 - \frac{1}{4}\,(x_0 - y_1 - (1-s))_+^2 \\ &\qquad\qquad\qquad - \frac{1}{4}\,(y_0 - x_1 - (1-s))_+^2.\end{aligned}$$

Analyzing the expression in the right-hand side we see that if (13) were false, then $\mathcal{N} < \mathcal{O}$ along with (9) would give

$$x_0 + x_1 - y_0 - y_1 < x_0 - y_1 - (1-s),$$

which is (13) in disguise. This contradiction shows that (13) is true. We now readily get (12) as a consequence of (13) and (9), and (14) is just a sum of (13) and (12).

To summarize, there are two major cases to consider: that where (11) holds true, and that where (12)– (14) hold true. Since in the second case we have $G(x_0, y_1, 1-s) = y_1(1-s)$ and $G(x_1, y_0, 1-s) = x_1(1-s)$, the proof of Lemma 5 will be complete once we establish the following claims.

Claim 1. *We have $f(x_0, x_1, x_1, x_0) \le 0.15$ for any $x_0, x_1 \ge 0$ with $s := 2(x_0 + x_1) \in [1/2, 1]$.*

Claim 2. *For real $x_0, x_1, y_0,$ and y_1, write $s := x_0 + x_1 + y_0 + y_1$ and let*

$$g(x_0, x_1, y_0, y_1) = \min\{0.15s^2, x_0y_0 + x_1y_1\} + (x_1 + y_1)(1-s) + 0.25(1-s)^2.$$

Then $g(x_0, x_1, y_0, y_1) \le 0.15$ whenever $x_0, x_1, y_0, y_1 \ge 0$ satisfy (14), and $s \le 1$.

Proof of Claim 1. As

$$\begin{aligned}f(x_0, x_1, x_1, x_0) &= \min\{0.15s^2, 2x_0x_1\} \\ &\qquad + G(x_0, x_0, 1-s) + G(x_1, x_1, 1-s) + 0.25(1-s)^2,\end{aligned}$$

and since $x_0 + x_1 = \frac{1}{2}s$ implies $2x_0x_1 \le \frac{1}{8}\,s^2 < 0.15s^2$, we have to show that

$$2x_0x_1 + G(x_0, x_0, 1-s) + G(x_1, x_1, 1-s) + 0.25(1-s)^2 \le 0.15. \qquad (16)$$

We distinguish three cases.

Case I: $\max\{x_0, x_1\} \leq \frac{1}{2}(1-s)$. In this case, from the definition of the function G, we have $G(x_0, x_0, 1-s) = x_0^2$ and $G(x_1, x_1, 1-s) = x_1^2$. Therefore, (16) reduces to

$$2x_0x_1 + x_0^2 + x_1^2 + 0.25(1-s)^2 \leq 0.15$$

or, equivalently,

$$0.25s^2 + 0.25(1-s)^2 \leq 0.15. \tag{17}$$

To show this we notice that our present assumption $\max\{x_0, x_1\} \leq \frac{1}{2}(1-s)$ yields $s = 2(x_0 + x_1) \leq 2 - 2s$, implying $s \leq \frac{2}{3}$. However, the largest value attained by the left-hand side of (17) in the range $\frac{1}{2} \leq s \leq \frac{2}{3}$ is easily seen to be $5/36 < 0.15$.

Case II: $\min\{x_0, x_1\} \geq \frac{1}{2}(1-s)$. In this case, by Lemma 4, we have $G(x_0, x_0, 1-s) \leq x_0(1-s) - 0.25(1-s)^2$ and $G(x_1, x_1, 1-s) \leq x_1(1-s) - 0.25(1-s)^2$. Consequently, the left-hand side of (16) is at most

$$\begin{aligned}
2x_0x_1 &+ x_0(1-s) + x_1(1-s) - 0.25(1-s)^2 \\
&\leq \frac{1}{2}(x_0 + x_1)^2 + (x_0 + x_1)(1-s) - 0.25(1-s)^2 \\
&= \frac{1}{8}s^2 + \frac{1}{2}s(1-s) - 0.25(1-s)^2 \\
&= -\frac{5}{8}\left(s - \frac{4}{5}\right)^2 + 0.15 \\
&\leq 0.15.
\end{aligned}$$

Case III: $x_0 \leq \frac{1}{2}(1-s) \leq x_1$ (the case $x_1 \leq \frac{1-s}{2} \leq x_0$ being symmetric). In this case $G(x_0, x_0, 1-s) = x_0^2$, while from Lemma 4 we have $G(x_1, x_1, 1-s) \leq x_1(1-s) - 0.25(1-s)^2$; thus, (16) reduces to

$$2x_0x_1 + x_0^2 + x_1(1-s) \leq 0.15$$

and, substituting $x_0 = \frac{1}{2}s - x_1$ and re-arranging the terms, to

$$\frac{1}{4}(2s^2 - 2s + 1) - \left(x_1 - \frac{1}{2}(1-s)\right)^2 \leq 0.15. \tag{18}$$

Observing that $2s^2 - 2s + 1$ is increasing for $s \geq 1/2$ (and recalling that $s \geq \frac{1}{2}$ by the assumptions of the claim), we conclude that if $s \leq \frac{2}{3}$, then the left-hand side or (18) does not exceed

$$\frac{1}{4}\left(2 \cdot \frac{4}{9} - 2 \cdot \frac{2}{3} + 1\right) = \frac{5}{36} < 0.15.$$

If, on the other hand, $s \geq \frac{2}{3}$, then we have

$$x_1 = \frac{1}{2}s - x_0 \geq \frac{1}{2}s - \frac{1}{2}(1-s) = s - \frac{1}{2} \geq \frac{1}{2}(1-s),$$

whence the left-hand side of (18) does not exceed

$$\frac{1}{4}\left(2s^2-2s+1\right)-\left(\left(s-\frac{1}{2}\right)-\frac{1}{2}\left(1-s\right)\right)^2=-\frac{7}{4}\left(s-\frac{5}{7}\right)^2+\frac{1}{7}<0.15.$$

□

Proof of Claim 2. Since replacing x_0 and y_0 with their average $(x_0+y_0)/2$ and, simultaneously, x_1 and y_1 with their average $(x_1+y_1)/2$, can only increase the value of g, and does not affect the validity of (14), we can assume that $y_0=x_0$ and $y_1=x_1$. Thus, we want to show that in the region defined by

$$x_0,x_1\ge 0,\ \ x_0+x_1\le 1/2,\ \text{ and } 3x_0+x_1\ge 1, \tag{19}$$

we have

$$g(x_0,x_1,x_0,x_1)\le 0.15.$$

Observing that

$$\begin{aligned} g(x_0,x_1,x_0,x_1)&=\min\{0.6(x_0+x_1)^2,x_0^2+x_1^2\}+0.25(1-2x_0-2x_1)(1-2x_0+6x_1)\\ &=\min\{0.6(x_0+x_1)^2,x_0^2+x_1^2\}+x_0^2-2x_0x_1-3x_1^2-x_0+x_1+0.25, \end{aligned}$$

the estimate to prove can be re-written as

$$\min\{u(x_0,x_1),v(x_0,x_1)\}\le -0.1,$$

where

$$u(x_0,x_1)=2x_0^2-2x_0x_1-2x_1^2-x_0+x_1$$

and

$$v(x_0,x_1)=1.6x_0^2-0.8x_0x_1-2.4x_1^2-x_0+x_1.$$

Conditions (19) determine on the coordinate plane (x_0,x_1) a triangle with the vertices at $(1/3,0)$, $(1/2,0)$, and $(1/4,1/4)$. If $\varphi:=(3-\sqrt{5})/2$, then the line $x_1=\varphi x_0$ splits this triangle into two parts: a smaller triangle $\mathfrak{T}$ which inherits the vertex $(1/4,1/4)$ of the original triangle, and a rectangle $\mathfrak{R}$ inheriting the vertices $(1/3,0)$ and $(1/2,0)$ of the original triangle. (We consider both $\mathfrak{T}$ and $\mathfrak{R}$ as closed regions, so that they intersect by a segment.) The reason to partition the large rectangle as indicated is that

$$\min\{u(x_0,x_1),v(x_0,x_1)\}=\begin{cases} u(x_0,x_1) & \text{if } (x_0,x_1)\in\mathfrak{T},\\ v(x_0,x_1) & \text{if } (x_0,x_1)\in\mathfrak{R}, \end{cases}$$

as one can easily verify; we therefore have to prove that $u(x_0,x_1)\le -0.1$ for all $(x_0,x_1)\in\mathfrak{T}$, and $v(x_0,x_1)\le -0.1$ for all $(x_0,x_1)\in\mathfrak{R}$.

To this end we observe that, as a simple computation shows, the only critical point of u is $(0.3,0.1)$, and the only critical point of v is $(0.35,0.15)$. Since the former point lies

on the line $3x_0 + x_1 = 1$, and the latter on the line $x_0 + x_1 = 1/2$, these points do not belong to the interiors of $\mathfrak{T}$ and $\mathfrak{R}$. Hence, the maxima of u on $\mathfrak{T}$, and of v on $\mathfrak{R}$, are attained on the boundary of these regions. To complete the proof we now observe that

I. if $1/3 \le x_0 \le 1/2$ and $x_1 = 0$, then

$$v(x_0, x_1) = 1.6x_0^2 - x_0 \le 1.6 \cdot \frac{1}{4} - \frac{1}{2} = -0.1$$

(as $1.6x_0^2 - x_0$ is an increasing function of x_0 on the interval $[1/3, 1/2]$);

II. if $x_0 + x_1 = 1/2$, then

$$u(x_0, x_1) = x_0^2 + x_1^2 - 0.25 \ge 0,$$

and

$$v(x_0, x_1) = 0.6(x_0 + x_1)^2 - 0.25 = -0.1.$$

III. if $3x_0 + x_1 = 1$, then

$$u(x_0, x_1) = -10x_0^2 + 6x_0 - 1 = -10(x_0 - 0.3)^2 - 0.1 \le -0.1;$$

if, in addition, $(x_0, x_1) \in \mathfrak{R}$, then

$$1 = 3x_0 + x_1 \le (3 + \varphi)x_0,$$

whence $x_0 \ge 1/(3 + \varphi) = (9 + \sqrt{5})/38$ and therefore

$$\begin{aligned} v(x_0, x_1) &= -17.6x_0^2 + 9.6x_0 - 1.4 \\ &\le -17.6 \cdot \left(\frac{9 + \sqrt{5}}{38}\right)^2 + 9.6 \cdot \frac{9 + \sqrt{5}}{38} - 1.4 \\ &= -0.1001\ldots \end{aligned}$$

(as $(9 + \sqrt{5})/38 > 3/11$, and $-17.6x_0^2 + 9.6x_0 - 1.4$ is a decreasing function of x_0 for $x_0 \ge 3/11$);

IV. if $x_1 = \varphi x_0$ and $(x_0, x_1) \in \mathfrak{T} \cap \mathfrak{R}$, then

$$\begin{aligned} u(x_0, x_1) = v(x_0, x_1) &= (2 - 2\varphi - 2\varphi^2)x_0^2 + (\varphi - 1)x_0 \\ &= 4(\sqrt{5} - 2)x_0^2 - \frac{\sqrt{5} - 1}{2}\, x_0, \end{aligned}$$

being a convex function of x_0, attains its maximum for a value of x_0 which is on the boundary of the triangle $\mathfrak{T} \cup \mathfrak{R}$. However, we have already seen that u and v do not exceed the value of -0.1 on the part of the boundary they are responsible for. $\square$

This finally completes the proof of Lemma 5, and thus the whole proof of Theorem 1.

$\square$

4. Proof of Theorem 2

As in the proof of Theorem 1, we use induction on $|A|$ and, with Lemma 1 in mind, assume that A is a set of integers. Again, for $i,j \in \{0,1\}$ we let $A_i := \{a \in A \colon a \equiv i \pmod 2\}$ and $A_{ij} := \{a \in A \colon a \equiv i+2j \pmod 4\}$, and write $m := |A|,\ m_i := |A_i|$, and $m_{ij} := |A_{ij}|$. Dividing through all elements of A by their greatest common divisor and replacing A with $-A$, if necessary, we can assume that

$$0 \le m_0 < m \text{ and } m_{00} \le m_{01}. \tag{20}$$

We want to show that $T(A,-A,A) \le 0.5m^2 + 0.5m$.

We distinguish two major cases, depending on which of m_0 and m_1 is larger.

Case I: $m_0 \ge m_1$. Since $a-b=2c$ implies that a and b are of the same parity, we have the decomposition

$$\begin{aligned} T(A,-A,A) &= T(A_1,-A_1,A) + T(A_0,-A_0,A_1) + T(A_0,-A_0,A_0) \\ &= T(A_1,-A_1,A) + T(A_{00},-A_{01},A_1) + T(A_{01},-A_{00},A_1) \\ &\qquad\qquad + T(A_0,-A_0,A_0) \end{aligned}$$

(for the second equality notice that $a_0 - b_0 = 2c_1$ with $a_0, b_0 \in A_0$ and $c_1 \in A_1$ implies that either $a_0 \in A_{00},\ b_0 \in A_{01}$, or $a_0 \in A_{01},\ b_0 \in A_{00}$). We estimate the first summand in the right-hand side trivially, and use the induction hypothesis (cf. (20)) for the last summand, and Corollary 2 for the remaining two summands; this gives

$$T(A,-A,A) \le m_1^2 + 2G(m_{00},m_{01},m_1) + \frac12 + \frac12\,m_0^2 + \frac12\,m_0. \tag{21}$$

Keeping in mind (20), we now consider three further subcases.

Subcase I.a: $\max\{m_{00},m_{01},m_1\} = m_1$. Using (21) and recalling that, by the assumption of Case I, we have $m_1 \le m_0 = m_{00}+m_{01}$, we get

$$\begin{aligned} T(A,-A,A) &\le m_1^2 + 2m_{00}m_{01} - \frac12\,(m_{00}+m_{01}-m_1)^2 + \frac12 + \frac12\,m_0^2 + \frac12\,m_0 \\ &= \frac12\,m_1^2 + 2m_{00}m_{01} + m_0m_1 + \frac12\,m_0 + \frac12 \\ &\le \frac12\,m_1^2 + \frac12\,m_{00}^2 + \frac12\,m_{01}^2 + m_{00}m_{01} + m_0m_1 + \frac12\,m \\ &= \frac12\,m^2 + \frac12\,m. \end{aligned}$$

Subcase I.b: $\max\{m_{00}, m_{01}, m_1\} = m_{01} \le m_{00} + m_1$. By (21), using the estimate $\frac{1}{2}\, m_0^2 \le m_{00}^2 + m_{01}^2$, we obtain

$$\begin{aligned} T(A,-A,A) &\le m_1^2 + 2m_{00}m_1 - \frac{1}{2}\,(m_{00}+m_1-m_{01})^2 + \frac{1}{2} + \frac{1}{2}\, m_0^2 + \frac{1}{2}\, m_0 \\ &\le \frac{1}{2}\, m_1^2 + m_{00}m_1 + \frac{1}{2}\, m_{00}^2 + \frac{1}{2}\, m_{01}^2 + m_{00}m_{01} + m_1 m_{01} + \frac{1}{2} + \frac{1}{2}\, m_0 \\ &\le \frac{1}{2}\, m^2 + \frac{1}{2} + \frac{1}{2}\, m_0 \\ &\le \frac{1}{2}\, m^2 + \frac{1}{2}\, m. \end{aligned}$$

Subcase I.c: $\max\{m_{00}, m_{01}, m_1\} = m_{01} \ge m_{00}+m_1$. In this case we have $G(m_{00}, m_{01}, m_1) \le m_{00}m_1$, and (21) along with $\frac{1}{2}\, m_1 < m_1 \le m_{01} - m_{00}$ give

$$\begin{aligned} T(A,-A,A) &\le m_1^2 + 2m_{00}m_1 + \frac{1}{2} + \frac{1}{2}\, m_0^2 + \frac{1}{2}\, m_0 \\ &= \frac{1}{2}\,(m_1+m_0)^2 + \frac{1}{2}\, m_1^2 - m_0m_1 + 2m_{00}m_1 + \frac{1}{2} + \frac{1}{2}\, m_0 \\ &= \frac{1}{2}\, m^2 + m_1 \left(\frac{1}{2}\, m_1 + m_{00} - m_{01} \right) + \frac{1}{2} + \frac{1}{2}\, m_0 \\ &< \frac{1}{2}\, m^2 + \frac{1}{2}\, m. \end{aligned}$$

Case II: $m_1 \ge m_0$. In this case we use the decomposition

$$\begin{aligned} T(A,-A,A) &= T(A_0,-A_0,A) + T(A_1,-A_1,A_0) + T(A_1,-A_1,A_1) \\ &= T(A_0,-A_0,A) + T(A_{10},-A_{10},A_0) + T(A_{11},-A_{11},A_0) \\ &\qquad + T(A_1,-A_1,A_1). \end{aligned}$$

Using the trivial bound m_0^2 for the first summand, applying Corollary 2 to estimate the second and third summands, and observing that $a_1 - b_1 = 2c_1$ $(a_1, b_1, c_1 \in A_1)$ implies that exactly one of a_1 and a_2 is in A_{10} and another is in A_{11}, we get

$$T(A,-A,A) \le m_0^2 + G(m_{10}, m_{10}, m_0) + G(m_{11}, m_{11}, m_0) + \frac{1}{2} + 2m_{10}m_{11}. \tag{22}$$

Since the right-hand side is symmetric in m_{10} and m_{11}, without loss of generality we assume that $m_{10} \le m_{11}$. Consequently, by the assumption of Case II, we have $m_0 \le m_1 \le 2m_{11}$, and to complete the proof we consider two subcases, according to whether the stronger estimate $m_0 \le 2m_{10}$ holds.

Subcase II.a: $m_0 \le 2m_{10}$. In this case, by (22) and Lemma 4, and in view of $2m_{10}m_{11} \le \frac{1}{2}(m_{10}+m_{11})^2 = \frac{1}{2}\,m_1^2$, we have

$$\begin{aligned} T(A,-A,A) &\le m_0^2 + \left(m_{10}m_0 - \frac{1}{4}\,m_0^2\right) + \left(m_{11}m_0 - \frac{1}{4}\,m_0^2\right) + \frac{1}{2} + \frac{1}{2}\,m_1^2 \\ &= \frac{1}{2}\,m_0^2 + m_0m_1 + \frac{1}{2}m_1^2 + \frac{1}{2} \\ &= \frac{1}{2}\,m^2 + \frac{1}{2}. \end{aligned}$$

Subcase II.b: $2m_{10} \le m_0 \le 2m_{11}$. Acting as in the previous subcase, but using the trivial estimate for the second summand in (22), we get

$$\begin{aligned} T(A,-A,A) &\le m_0^2 + m_{10}^2 + \left(m_{11}m_0 - \frac{1}{4}\,m_0^2\right) + \frac{1}{2} + 2m_{10}m_{11} \\ &\le \frac{3}{4}\,m_0^2 + m_{10}^2 + m_{11}m_0 + \frac{3}{2}\,m_{10}m_{11} + \frac{1}{4}\,m_{10}^2 + \frac{1}{4}\,m_{11}^2 + \frac{1}{2} \\ &= \frac{1}{2}(m_0+m_{10}+m_{11})^2 - \frac{1}{4}\,(m_{01}+m_{11}-m_0)(m_0+m_{11}-3m_{10}) + \frac{1}{2} \\ &\le \frac{1}{2}\,m^2 + \frac{1}{2}, \end{aligned}$$

the last inequality following from $m_{01}+m_{11}-m_0 \ge 0$ and $m_0+m_{11}-3m_{10} \ge 0$, by the present subcase assumptions.

This completes the proof of Theorem 2.

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